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Asymptotic behaviour of the perturbation expansion in $n = 0$ field theories

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Abstract. We develop a method for studying the high-order behaviour of the perturbation expansion in theories in which the number of field components, n, is taken to zero. This procedure is illustrated on the field theory formulation of the percolation problem, which can be considered as the $n = 0$ limit of the $(n+1)$-state Potts model. The saddle points controlling the asymptotic behaviour are labelled by an integer $r = 1, 2, \ldots, n$ for positive integer n, but after continuation to $n = 0$ the dominant contribution effectively comes from the saddle point with $r = \infty$. The perturbation expansion is found to be oscillatory at large orders and its behaviour is calculated.

1. Introduction

In the field theoretic study of critical phenomena the Hamiltonian is constructed by adding together all interaction terms allowed by the symmetry of the system. This Landau-Ginzburg-Wilson Hamiltonian will therefore have interactions cubic in the fields (the term cubic as used here means the product of three fields) unless this is forbidden by the symmetry. If cubic interactions are present then quartic and other higher-order terms are naively irrelevant as far as critical behaviour is concerned and are in the first instance ignored. In the renormalisation group (RG) approach (Amit 1978) one now (a) looks for fixed points of the theory with only cubic interactions present and then (b) checks that the omission of the quartic and higher-order terms was justified by calculation of their anomalous dimension near the fixed point(s) obtained in (a).

However, this approach is clearly not valid for models with conventional cubic interactions, since such field theories have Hamiltonians which are not bounded below and higher-order terms have to be added to the Hamiltonian simply to make the theory exist. Exceptions to this may occur when unphysical limits are taken, such as when the number of field components, n, is taken to be zero or negative (which will be the case of interest to us here) or when no such limit is taken but the coupling is pure imaginary. This distinction between conventional cubic theories which do not exist and those which do is not obvious from low-order perturbation theory, but if one just goes ahead and applies the RG procedure outlined above, the difference does seem to show up. Firstly the structure of the fixed point equation to lowest order in the ε expansion is such that fixed points are more likely for unphysical values of n or physical values of n but with an imaginary coupling constant, indicating that these theories are more likely to have continuous phase transitions. Secondly, even if a fixed point exists to lowest order for conventional theories (for example, the three-state Potts model), it is found that quartic interactions tend to become relevant as ε increases (Amit *et al* 1977). This points towards the necessity of including quartic interactions to stabilise the theory. On the other hand the theories with unphysical n or imaginary coupling constant, for which the relevance of quartic interactions has been investigated, do not seem to have this tendency (Amit *et al* 1977, Elderfield and McKane 1978, Kirkham and Wallace 1979).

The methods developed by Lipatov (1977a, b) and Brézin *et al* (1977) are clearly suited to the study of whether or not a particular cubic theory exists in its own right without the need for stabilising terms. The application of these methods is straightforward in the case of conventional theories where n is a positive integer (McKane 1979). It is found, as expected, that the theory is unstable when the coupling constant is real. This leads to a perturbation expansion which is divergent and non-oscillatory at large order. If the coupling constant is pure imaginary, however, there is no instability and the perturbation expansion is Borel summable. This is the situation in the problem of Yang-Lee edge singularities (Fisher 1978, Kirkham and Wallace 1979) and the information obtained from the asymptotic behaviour of the perturbation expansion has led to improved estimates for universal quantities (de Alcantara Bonfim *et al* 1980, 1981).

But it is precisely the set of theories that are of the most physical interest—those with n not an integer, which have some chance of being stable—that have not yet yielded to analysis†. It is the purpose of this paper to set out a method for determining the asymptotic behaviour of the perturbation expansion for these theories, and in particular theories with $n=0$. We restrict ourselves to the study of the percolation problem, which is the $n=0$ limit of the $(n+1)$-state Potts model, for three reasons. Firstly, it is a comparatively simple $n=0$ theory and is therefore useful for developing techniques which can then be applied to more complicated systems, such as spin glasses. Secondly, it is of considerable interest in its own right and knowledge of the asymptotic behaviour should lead to improved estimates for critical exponents. Thirdly, the only attempt so far to understand the asymptotic behaviour of $n=0$ theories with cubic interactions was made on this problem by Houghton *et al* (1978), to be referred to as HRW. They attempted to circumvent difficulties associated with the $n=0$ limit by reformulating the theory so as to avoid this limit. This reformulation is special to the percolation problem and cannot be applied to other types of interaction. Originally our intention was to check our method, in which the limit $n\to 0$ is taken explicitly, with the result of HRW for the percolation problem. Having checked our method we could then go on to apply it to other situations with confidence. However, as explained below and in more detail in § 2, we do not agree with HRW and have enough confidence in our method to question their result.

The determination of the asymptotic behaviour of the perturbation expansion in field theory begins with the observation that the imaginary part of the Green functions, generated for values of the coupling constant for which the theory is unstable, can be calculated by studying the classical solutions of the field equations (the instantons) of the theory (Itzykson and Zuber 1980). Having obtained the imaginary part of the Green function for a value of the coupling constant for which the theory is unstable, the asymptotic behaviour of the perturbation expansion for a value of the coupling

† With the exception of the work of Houghton *et al* on the percolation problem which is discussed further below.

constant for which the theory is stable is obtained by a dispersion relation. If one applies these ideas to a wide class of $n \to 0$ theories then one finds that the instanton solution of least action, which should give the dominant contribution to the asymptotic behaviour, has an action proportional to n. This means that the Kth-order term in the perturbation expansion for large K, behaves like n^{-K} as $n \to 0$. This is clearly nonsense and yet the solution with least action certainly gives this result. As we have mentioned already, the only progress made on this point so far has been by HRW, who did not try to understand this difficulty further but instead tried to reformulate the percolation problem to avoid the difficulty and in this way found a well defined expression for the Kth order of the perturbation expansion for large K. Moreover they found a series which was oscillatory for large K, implying that the theory is stable as $n \to 0$. Here we will show how one can recover the results of HRW for the percolation problem without the reformulation and by instead explicitly taking the limit $n \to 0$. We will then go on to argue that this is not a correct way of handling the limit; a more careful treatment gives a different form for the asymptotic behaviour, although the series is still found to be oscillatory. Furthermore, this treatment should be applicable to other $n = 0$ theories.

The layout of the paper is as follows. In § 2 we introduce the $(n+1)$-state Potts model as a field theory and calculate the asymptotic behaviour of the perturbation expansion for the zero-dimensional ($d = 0$) version. This has all the structure needed to illustrate the problems associated with the $n = 0$ limit. We obtain the $d = 0$ version of the result of HRW and comment on the validity of this approach. In § 3 we reconsider the asymptotics of the zero-dimensional percolation field theory but with a more careful consideration of the $n = 0$ limit and in § 4 we extend this approach to the field theory in $6 - \varepsilon$ dimensions. Various observations and general comments are made in § 5. An appendix contains some added technical details relevant to the discussion in §§ 2 and 3.

2. Illustration of the problem

In this section we will obtain the asymptotic behaviour of the perturbation expansion for the partition function of the $d = 0$ field theory with the symmetry of the $(n+1)$-state Potts model. This will enable us to go on to illustrate the problems which occur when one tries to perform such analyses with $n = 0$.

The $(n+1)$-state Potts model in d dimensions can be written as a field theory with a Hamiltonian of the form (Zia and Wallace 1975)

$$\mathscr{H}(\phi_i) = \int d^dx[\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i) + \tfrac{1}{2}r_0\phi_i\phi_i + (g/3)\rho_{ijk}\phi_i\phi_j\phi_k + O(\phi^4)] \tag{2.1}$$

where the summation convention is assumed, $i = 1, 2, \ldots, n$ and where

$$\rho_{ijk} = \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha. \tag{2.2}$$

Here $\{e_i^1, e_i^2, \ldots, e_i^{n+1}\}$ is a set of $(n+1)$ n-component vectors specifying the vertices of a hypertetrahedron in $(n+1)$ dimensions. They satisfy

$$\sum_{i=1}^{n} e_i^\alpha e_i^\beta = (n+1)\delta^{\alpha\beta} - 1, \qquad \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha = (n+1)\delta_{ij}, \qquad \sum_{\alpha=1}^{n+1} e_i^\alpha = 0. \tag{2.3}$$

The quartic and higher-power terms will be omitted from (2.1) from now on for the reasons discussed in § 1. In this paper we will be particularly interested in the cases $d=0$ (for illustrative purposes) and $d=6-\varepsilon$ (this is analytically tractable and has applications to the ε expansion) but the methods developed hold for any $d<6$, in particular $d=3$. In the rest of this section and the whole of § 3 we will be exclusively concerned with the $d=0$ theory.

In zero dimensions there are no spatial variables for the fields to depend upon and thus

$$\mathscr{H}(\phi_i)|_{d=0}=\tfrac{1}{2}\phi_i\phi_i+(g/3)\rho_{ijk}\phi_i\phi_j\phi_k. \tag{2.4}$$

For simplicity, we have chosen the normalisation so that the coefficient of $\phi_i\phi_i$ is $\frac{1}{2}$. The partition function is then an n-fold integral:

$$Z_n(g)=\int \mathrm{d}\boldsymbol{\phi}\ \exp-[\tfrac{1}{2}\phi_i\phi_i+(g/3)\rho_{ijk}\phi_i\phi_j\phi_k] \tag{2.5}$$

where $\boldsymbol{\phi}=(\phi_1,\phi_2,\ldots,\phi_n)$. Expanding in g we see that the series takes the form

$$Z_n(g)=(2\pi)^{n/2}\left(1+\sum_{K=1}^{\infty} Z_K(n)g^{2K}\right). \tag{2.6}$$

Our aim in this section is to find the form of $Z_K(n)$ for K large, initially with n a positive integer but subsequently with $n\to 0$.

Clearly the integral in (2.5) diverges at large ϕ_i for real g. This is reflected in the fact that $Z_n(g)$ develops an imaginary part for real g. To calculate it, let $\phi_i=-(1/g)u_i$, then

$$Z_n(g)=g^{-n}\int \mathrm{d}\boldsymbol{u}\ \exp(1/g^2)(-\tfrac{1}{2}u_iu_i+\tfrac{1}{3}\rho_{ijk}u_iu_ju_k). \tag{2.7}$$

For small g we may evaluate this integral by the method of steepest descent. Saddle points are found by solving

$$u_i=\rho_{ijk}u_ju_k,\qquad i=1,2,\ldots,n. \tag{2.8}$$

The general solution to (2.8) may be written as

$$u_i^{(r)}=\frac{1}{(n+1)(n+1-2r)}\sum_{\alpha=1}^{n+1} e_i^{\alpha}a_{(r)}^{\alpha} \tag{2.9}$$

where $r=0,1,2,\ldots,n+1$ is an index classifying the solutions in the sense that $a_{(r)}^{\alpha}$ is an $(n+1)$-component vector with r entries equal to unity and $(n+1-r)$ equal to zero. Actually because of the last relation in (2.3) it is easy to see that the solutions characterised by $r'=(n+1-r)$ are not distinct from those characterised by r. Also for odd values of n the solutions with $r=\frac{1}{2}(n+1)$ do not exist since the normalisation factor in (2.9) diverges. In summary then, $r=0,1,2,\ldots,\frac{1}{2}n$ for n even and $r=0,1,2,\ldots,\frac{1}{2}(n-1)$ for n odd gives us all the solutions. For each r there are ${}^{n+1}C_r$ solutions but they all give the same value for the exponent in (2.7), that is, the 'action' only depends on r:

$$A^{(r)}\equiv\tfrac{1}{2}u_i^{(r)}u_i^{(r)}-\tfrac{1}{3}\rho_{ijk}u_i^{(r)}u_j^{(r)}u_k^{(r)}=\frac{r(n+1-r)}{6(n+1)^2(n+1-2r)^2},$$

$$r = 0, 1, 2, \ldots, [n/2] \tag{2.10}$$

where the notation $[n/2]$ means the largest integer less than or equal to $n/2$. To find the factor multiplying the exponent one needs to find the eigenvalues of

$$\begin{aligned} M_{ij}^{(r)} &\equiv \frac{\partial^2}{\partial u_i\, \partial u_j}\left(-\tfrac{1}{2}u_k u_k + \tfrac{1}{3}\rho_{klm}u_k u_l u_m\right)_{u=u^{(r)}} \\ &= \left(-\delta_{ij} + 2\rho_{ijk}u_k^{(r)}\right) \\ &= \frac{1}{(n+1-2r)}\left(-(n+1)\delta_{ij} + 2\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha a_{(r)}^\alpha\right). \end{aligned} \tag{2.11}$$

If $r = 0$, $M_{ij}^{(r)} = -\delta_{ij}$ and so all the eigenvalues are equal to -1. If $r = 1$, with $a_{(r)}^\alpha = \delta^{\alpha 1}$, say, then the eigenvectors are e_i^1 and $\{e_i^1 + \ldots + e_i^s + (n+1-s)e_i^{s+1}\colon\ s = 1, \ldots, n-1\}$ with eigenvalues 1 and $-(n+1)/(n-1)$ respectively. Thus for $r = 1$ there is one eigenvalue equal to 1 and $(n-1)$ equal to $-(n+1)/(n-1)$. The case of general r is given below.

Suppose $a_{(r)}^\alpha = \delta^{\alpha 1} + \delta^{\alpha 2} + \ldots + \delta^{\alpha r}$:

eigenvector $e_i^1 + \ldots + e_i^r$ has eigenvalue 1;

eigenvectors $e_i^1 + \ldots + e_i^s - se_i^{s+1}$ $(s = 1, 2, \ldots, r-1)$ have

eigenvalue $(n+1)/(n+1-2r)$; (2.12)

eigenvectors $e_i^1 + \ldots + e_i^{r+s-1} + (n-r-s+2)e_i^{r+s}$ $(s = 1, 2, \ldots, n-r)$ have

eigenvalue $-(n+1)/(n+1-2r)$.

Thus there is one eigenvalue 1, $(n-r)$ eigenvalues $-(n+1)/(n+1-2r)$ and $(r-1)$ eigenvalues $(n+1)/(n+1-2r)$. Clearly while the eigenvectors depend on the choice of $a_{(r)}^\alpha$ the eigenvalues do not.

Using the above information we may write down an expression for $Z_n(g)$ when g is small:

$$\begin{aligned} Z_n(g) = (2\pi)^{n/2}[1 + \mathrm{O}(g^2)] + \sum_{r=1}^{[n/2]} {}^{n+1}C_r \left(\frac{2\pi}{-1}\right)^{1/2}\left(\frac{2\pi(n+1-r)}{(n+1)}\right)^{(n-r)/2} \\ \times\left(\frac{2\pi(n+1-r)}{-(n+1)}\right)^{(r-1)/2} \exp\left(\frac{-r(n+1-r)}{6(n+1)^2(n+1-2r)^2}\frac{1}{g^2}\right)[1 + \mathrm{O}(g^2)]. \end{aligned} \tag{2.13}$$

The first term on the RHS of (2.13) is the $r = 0$ contribution. It gives the usual perturbation expansion about $\phi_i = 0$. The contributions from the non-trivial saddle points admit the possibility of imaginary parts being generated. In our case there are $1 + (r-1) = r$ positive eigenvalues† for the solution characterised by r which means that odd values of r give pure imaginary contributions.

Before proceeding further with the analysis let us comment on the basis of the steepest descent calculation. Since (2.5) does not exist as it stands, the correct procedure is of course to define analogous partition functions where the integration over ϕ_i is along contours in the complex plane, the contours being chosen so that the functions

† With our definition of $M_{ij}^{(r)}$, positive eigenvalues lead to the generation of imaginary parts.

are well defined for $\mathrm{Re}\, g>0$. By rotating the contours one can define these functions for other values of g by analytic continuation. This procedure has been discussed in detail for the single-component ϕ^3 theory (McKane 1979). There it was shown that there are essentially two choices for the partition function, $Z^{(2)}(g)$ and $Z^{(3)}(g)$ (in the slightly modified notation of that paper), with $Z^{(2)*}(g)=Z^{(3)}(g)$. The function $Z(g)$ is taken to equal $Z^{(2)}(g)$ for $0<\arg<\pi$ and $Z^{(3)}(g)$ for $0>\arg g>-\pi$, with a branch cut for all real g. Since these are really functions of g^2 one either works with $Z^{(2)}(g^2)$, $0<\arg g^2<2\pi$, or with $Z^{(3)}(g^2)$, $0>\arg g^2>-2\pi$. In this more rigorous approach the factors of i in the steepest descent calculation came from the paths of steepest descent going off into the complex plane. However, experience with the single-component ϕ^3 theory and other toy systems shows that the straightforward approach adopted in our case gives the correct result apart from (i) ambiguities in the sign of factors of i, and (ii) possible factors of $\frac{1}{2}$ coming from the fact that for some of the saddle points only $\frac{1}{2}$ the Gaussian integral coming from perturbations about the saddle point may contribute. This is rather fortunate since contours in $\mathbb{C}^n$ are not easily visualised and we choose to determine the signs of the factors of i and the factors of $\frac{1}{2}$ in a different way. This is discussed in detail in the appendix where it is shown that

$$Z_n(g)=(2\pi)^{n/2}[1+\mathrm{O}(g^2)]+\tfrac{1}{2}\sum_{r=1}^{[n/2]}{}^{n+1}C_r(\pm\mathrm{i})^r(2\pi)^{n/2}\left(\frac{(n+1-2r)}{(n+1)}\right)^{(n-1)/2}$$
$$\times\exp\left(\frac{-r(n+1-r)}{6(n+1)^2(n+1-2r)^2}\frac{1}{g^2}\right)[1+\mathrm{O}(g^2)] \qquad (2.14)$$

where the plus sign is taken if $\arg g=0$ and the minus sign if $\arg g=\pi$.

The contribution from all saddle points has been explicitly evaluated and displayed in (2.14) since we will need this information when we consider the continuation to $n=0$. However, if we are interested in positive integer n, then the leading contribution to $\mathrm{Im}\, Z_n(g)$ comes from the $r=1$ solution only. This is because $A^{(r)}$ given by (2.10) is a monotonically increasing function of r in the range of interest and so all other contributions are exponentially smaller. Therefore,

$$\mathrm{Im}\, Z_n(g)=\pm\tfrac{1}{2}(n+1)(2\pi)^{n/2}\left(\frac{(n-1)}{(n+1)}\right)^{(n-1)/2}$$
$$\times\exp\left(\frac{-n}{6(n+1)^2(n-1)^2}\frac{1}{g^2}\right)[1+\mathrm{O}(g^2)] \qquad (2.15)$$

where the exponentially smaller terms have been omitted and where the upper sign corresponds to $\arg g=0$ and the lower sign to $\arg g=\pi$.

Having obtained $\mathrm{Im}\, Z_n(g)$, the second step in determining the asymptotic behaviour of the perturbation expansion is to use the analytic structure of $Z_n(g^2)$ to write a dispersion relation relating $\mathrm{Re}\, Z_n(g^2)$ for $g^2<0$ to $\mathrm{Im}\, Z_n(g^2)$ for $g^2>0$. Since $Z_n(g^2)$ is analytic apart from a branch cut along the positive real g^2 axis, we consider the contour shown in figure 1. Then

$$Z_n(g)=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{Z_n(g'^2)\,\mathrm{d}g'^2}{g'^2-g^2}, \qquad g^2<0. \qquad (2.16)$$

But $Z_n(|g^2|)\sim|g^2|^{-n/6}$ as $|g|\to\infty$ and so the contour at infinity can be discarded and we are left with

$$Z_n(g^2)=\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\, Z_n(g'^2;\arg g'^2=0)\,\mathrm{d}g'^2}{g'^2-g^2}. \qquad (2.17)$$

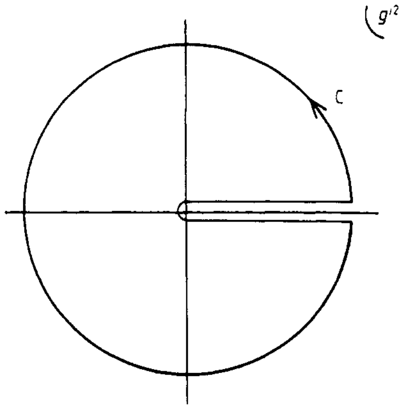

Figure 1. Contour involved in the dispersion relation for positive integer n.

Identifying terms in the perturbation expansion using (2.6) (which of course has the same form for $g^2<0$) we find

$$Z_K(n)=\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, Z_n(g'^2;\, \arg g'^2=0)\, \mathrm{d}g'^2}{(g'^2)^{K+1}}. \tag{2.18}$$

This result is true for all K, but we only know $\mathrm{Im}\, Z_n(g'^2)$ for small g'^2 which means we can determine $Z_K(n)$ for large K. Substituting (2.15) into (2.18) gives

$$Z_K(n)=\frac{1}{2\pi}(n+1)(2\pi)^{n/2}\left(\frac{n-1}{n+1}\right)^{(n-1)/2}$$
$$\times\left(\frac{6(n+1)^2(n-1)}{n}\right)^K K!K^{-1}\left[1+\mathrm{O}\left(\frac{1}{K}\right)\right]. \tag{2.19}$$

In the appendix we evaluate $Z_K(2)$ and $Z_K(3)$ directly and find agreement with this result. From (2.6) and (2.19) we see as expected that the perturbation expansion is non-oscillatory for large K when $g^2>0$ and oscillatory when $g^2<0$.

We now go on to investigate the limit $n\to 0$. A glance at (2.19) reveals a problem: $Z_K(n)\sim n^{-K}$ as $n\to 0$. Since it is obvious from the structure of the perturbation expansion that $Z_K(n)$ exists as $n\to 0$ (in fact it is clear that $Z_K(n)\sim n$ as $n\to 0$), clearly this saddle point cannot describe the asymptotic behaviour of the $n=0$ theory. So we may ask what saddle point does dominate and how is the previously dominant $r=1$ saddle point excluded? We have been able to make progress on this point because in addition to calculating the action for all r, we have calculated the prefactors in (2.14) and thus in (2.19) for all r as well. Let us first give an argument which leads to $Z_K(n)\sim n$ for $n\to 0$ and moreover agrees with HRW when extended to field theory.

In the limit $n\to 0$ the action of the solution changes sign: from (2.10)

$$\frac{1}{g^2}\lim_{n\to 0} A^{(r)}=\frac{-r(r-1)}{6(1-2r)^2 g^2}\equiv\frac{a^{(r)}}{g^2}. \tag{2.20}$$

Whereas $A^{(r)}/g^2\geqslant 0$ for positive integer n and $g^2>0$, the analogous $n\to 0$ quantity, $a^{(r)}/g^2$, satisfies $a^{(r)}/g^2\leqslant 0$ for $g^2>0$. This reflects the fact that the $n=0$ theory does not exist when g is pure imaginary ($g^2<0$) but is well defined when g is real ($g^2>0$). Thus in order to generate an imaginary part for $Z_n(g)$ and hence $Z_K(n)$ for $n\to 0$, K

large, we have to work with $g^2<0$. In this case the expression on the RHS of (2.20) is a monotonically increasing function of r for $r\geqslant 1$, and the low r solutions should dominate.

The $r=0$ solution $\phi_i=0$ again gives the usual perturbation expansion (2.6). The $r=1$ solution has an action which goes like n for small n which naively leads to the problem discussed above. However, if the prefactors in either (2.13) or (2.14) are examined, one sees that instead of picking up one factor of i from one positive eigenvalue, n factors of i are picked up—all n eigenvalues are positive as $n\to 0$. Thus the prefactor contains $(\mathrm{i})^n\to 1$ as $n\to 0$†. Therefore, the $r=1$ solution does not contribute to the imaginary part of $Z_n(g)$. Next, looking at the $r=2$ solution, we see that an imaginary part is generated. Further the action is non-zero as $n\to 0$ and so no n^{-K} divergence is encountered. Also since ${}^{n+1}C_2\sim n/2$ as $n\to 0$ we expect $Z_K(n)\sim n$ as required. Since the $r=3, 4, \ldots$ solutions give contributions exponentially smaller than the $r=2$ solution, we have

$$\operatorname{Im} Z_n(g)=\pm\frac{n}{4\sqrt{3}}\exp\left(-\frac{1}{27|g^2|}\right)[1+\mathrm{O}(g^2)]+\mathrm{O}(n^2), \qquad g \text{ pure imaginary.} \tag{2.21}$$

Assuming we can write a dispersion relation as in the case of positive integer n (but with the contour now going around the real negative g^2 axis), we would expect

$$Z_K(n)=\frac{n}{4\sqrt{3}}(-27)^K K!\,K^{-1}\left[1+\mathrm{O}\left(\frac{1}{K}\right)\right]+\mathrm{O}(n^2) \tag{2.22}$$

for large K and $n\to 0$. When this approach is extended to field theory in $d=6-\varepsilon$ dimensions it gives the results of HRW. The result (2.22) has all the features we expect and desire but we will now go on to argue that it is incorrect.

Suppose n is even. The $r=n/2$ solutions exist and have an action proportional to n with no factor of i multiplying the exponential. From (2.14) it can be seen that these solutions give a factor of $-\frac{1}{2}+\mathrm{O}(g^2)$ to $Z_n(g)$ as $n\to 0$. The solutions with $r=n/2-1$ give a contribution equal to (2.21) up to a factor of $\frac{1}{2}$ coming from the fact that ${}^{n+1}C_{n/2-1}\sim n/4$ as $n\to 0$ as compared with ${}^{n+1}C_2\sim n/2$ as $n\to 0$. Indeed, as $n\to 0$ the $r'=n/2+1-r$ solutions yield essentially the same results as the solution characterised by r. Thus we should add the $r=n/2-1$ result to the $r=2$ one in (2.21). This will change the value of the prefactor. But why not add the $r=n/4-1$ and the $r=n/4+2$ solutions if n is divisible by 4, since they give the same action as $n\to 0$? If n is odd, even less acceptable behaviour is found, with the $r=n/2-\frac{1}{2}$, $n/2-\frac{3}{2}, \ldots$ solutions giving terms not proportional to n for small n and with the action of the $r=n/2-\frac{1}{2}$ solutions being less than the $r=2$ action as $n\to 0$. Clearly this is a very ill defined procedure with the prefactor of (2.21) indeterminate, and more alarmingly the fundamental result $Z_n(g)=1+\mathrm{O}(n)$, for $n\to 0$, violated. We claim that a more controlled procedure for taking n to zero is required and in particular that one must take seriously the fact that the sum in (2.14) contains $[n/2]$ terms as $n\to 0$. HRW did note in their paper that their results could be obtained by setting $(r, n)=(2,0)$ or $(-1,0)$ in the formula for the action. It is not clear to us if their method picks up these solutions because the $r=1$ and $r=0$ solutions give real contributions or because these contributions are independent of g^2 when $n=0$. We will not pursue these points any further here but instead go on to discuss a more satisfactory continuation to $n=0$.

† It might be argued that it is more correct to say that as $g^2\to -g^2$ the eigenvalues coming from the exponent in (2.7) change sign, thus they all become negative as $n\to 0$ and no imaginary part is generated at all.

3. The $n=0$ limit of the zero-dimensional field theory

The steepest descent evaluation of the zero-dimensional partition function, $Z_n(g)$, for small g, leads to a sum over the contributions from the various saddle points with different actions. The number of terms in the sum becomes a meaningless concept as $n\to 0$ and in § 2 it was argued that it was therefore not correct to pick out one term in the series as being the most dominant as $n\to 0$. Thus one should continue the sum of terms rather than any individual term to $n=0$. The usual way of doing this is to write an integral representation for the sum with n appearing as a parameter. The precise method we adopt is one used in some many-body calculations, namely we write

$$\sum_{r=1}^{n} f(r,n;g^2)=\oint_C \frac{f(z,n;g^2)\,\mathrm{d}z}{\mathrm{e}^{2\pi\mathrm{i}z}-1} \tag{3.1}$$

where C is a contour surrounding the positive integers $1,2,\ldots,n$ (shown in figure 2(a)) and $f(z,n;g^2)$ is analytic within the contour C. If $f(z,n;g^2)$ is analytic in the larger region $\delta\leqslant \mathrm{Re}(z)\leqslant n+\delta$, $-Y_-\leqslant \mathrm{Im}(z)\leqslant Y_+$, with $0<\delta<1$ and Y_+, $Y_->0$, then we can deform the contour as shown in figure 2(b) and obtain

$$\sum_{r=1}^{n} f(r,n;g^2)=\left(\int_{\delta+\mathrm{i}Y_+}^{\delta-\mathrm{i}Y_-}+\int_{\delta-\mathrm{i}Y_-}^{n+\delta-\mathrm{i}Y_-}+\int_{n+\delta-\mathrm{i}Y_-}^{n+\delta+\mathrm{i}Y_+}+\int_{n+\delta+\mathrm{i}Y_+}^{\delta+\mathrm{i}Y_+}\right)\frac{f(z,n;g^2)\,\mathrm{d}z}{\mathrm{e}^{2\pi\mathrm{i}z}-1}. \tag{3.2}$$

Let us denote the four integrals on the RHS by I_1, I_2, I_3 and I_4 respectively. Changing variables to $z'=z-n$ in I_3 we see that

$$I_1+I_3=\int_{\delta-\mathrm{i}Y_-}^{\delta+\mathrm{i}Y_+}\frac{[f(z+n,n;g^2)-f(z,n;g^2)]\,\mathrm{d}z}{\mathrm{e}^{2\pi\mathrm{i}z}-1}. \tag{3.3}$$

Taking $n\to 0$, this gives

$$I_1+I_3=n\int_{\delta-\mathrm{i}Y_-}^{\delta+\mathrm{i}Y_+}\frac{f'(z,0;g^2)\,\mathrm{d}z}{\mathrm{e}^{2\pi\mathrm{i}z}-1}+\mathrm{O}(n^2) \tag{3.4}$$

assuming $\lim_{n\to 0} f'(z,n;g^2)$ exists and equals $f'(z,0;g^2)$. Integrating (3.4) by parts, the integration of the exact differential is found to equal exactly $-(I_2+I_4)$ to $\mathrm{O}(n)$, and finally one finds

$$I_1+I_2+I_3+I_4=2\pi\mathrm{i}n\int_{\delta-\mathrm{i}Y_-}^{\delta+\mathrm{i}Y_+}\frac{f(z,0;g^2)\,\mathrm{e}^{2\pi\mathrm{i}z}}{(\mathrm{e}^{2\pi\mathrm{i}z}-1)^2}\,\mathrm{d}z+\mathrm{O}(n^2). \tag{3.5}$$

If the function $f(z,0;g^2)$ is such that we can take Y_+, $Y_-\to\infty$ then (3.5) may be expressed as

$$\sum_{r=1}^{n} f(r,n;g^2)=-\frac{\pi\mathrm{i}n}{2}\int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty}\frac{f(z,0;g^2)\,\mathrm{d}z}{\sin^2\pi z}+\mathrm{O}(n^2). \tag{3.6}$$

We now apply (3.6) to the expression (2.14) for $Z_n(g)$. Several points should be made at this stage. Firstly, the sum in (2.14) extends to $[n/2]$, not n. For simplicity we suppose n to be even so that $[n/2]=n/2$ is an integer and (3.6) can be used. A more general treatment in which this restriction is not necessary is given in the appendix. Secondly, inspection of the analytic properties of the summand in (2.14) shows that the formula (3.1) is applicable and also that the distortion of the contour from that shown in figure 2(a) to that shown in figure 2(b) is allowed. Thirdly, we need to take $g^2<0$ in order to generate an imaginary part for $n=0$ as discussed in § 2; the correct signs for the factors of i depend on whether $\arg g=\pi/2$ or $\arg g=-\pi/2$ and are determined in the appendix. Fourthly, the first term in (2.14) is perturbative and real

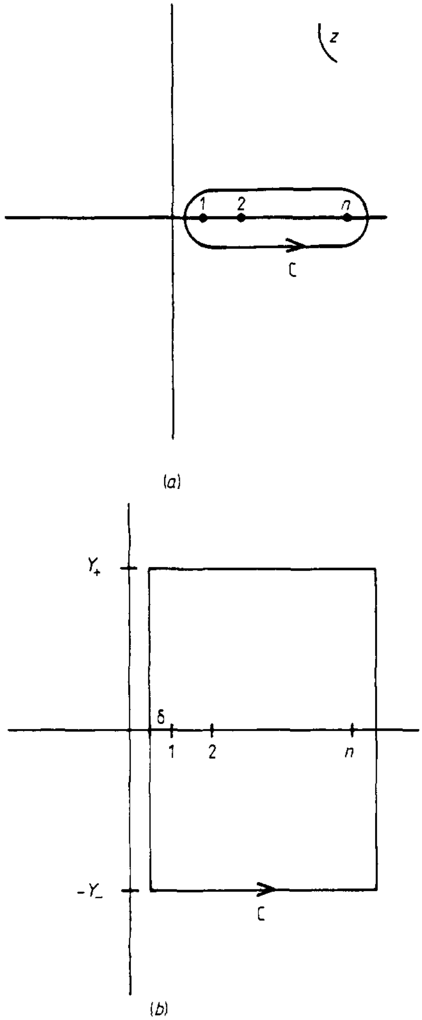

Figure 2. (*a*) Contour surrounding the integers $r=1,2,\ldots,n$. (*b*) Distortion of contour in (*a*) to facilitate continuation to $n=0$.

and presents no difficulty when continuing to $n=0$. It has the form $1+\mathrm{O}(n)$. Bearing all these points in mind, (2.14) and (3.6) lead to

$$\begin{aligned}
&\lim_{n\to 0}\frac{1}{n}[Z_n(g)-1]\\
&\quad=\frac{1}{2}\left(-\frac{\mathrm{i}\pi}{4}\right)\int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty}\frac{\mathrm{d}z\exp(\mp\mathrm{i}\pi z/2)(1-2z)^{-1/2}}{(\sin^2\pi z)z(1-z)\pi\operatorname{cosec}\pi z}\\
&\qquad\times\exp\left(\frac{z(1-z)}{6(1-2z)^2}\frac{1}{|g|^2}\right)[1+\mathrm{O}(g^2)]\\
&\quad=-\frac{\mathrm{i}}{8}\exp\left(-\frac{1}{24|g|^2}\right)\int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty}\frac{\mathrm{d}z\exp(\mp\mathrm{i}\pi z/2)}{z(1-z)(1-2z)^{1/2}\sin\pi z}\\
&\qquad\times\exp\left(\frac{1}{24(1-2z)^2}\frac{1}{|g|^2}\right)[1+\mathrm{O}(g^2)]
\end{aligned}\tag{3.7}$$

where only non-perturbative contributions have been kept and where the upper sign is taken if $\arg g = -\pi/2$ and the lower sign if $\arg g = -\pi/2$. Taking $\delta = \frac{1}{2}$ without loss of generality and writing $z = \frac{1}{2} + \mathrm{i}y$, one finds that the real parts of (3.7) are equal but that the imaginary parts are equal in magnitude but opposite in sign. Specifically

$$\lim_{n\to 0} \frac{1}{n} \operatorname{Im} Z_n(g)$$
$$= \mp \frac{1}{8\sqrt{2}} \exp\left(-\frac{1}{24|g|^2}\right) \int_0^\infty \frac{\mathrm{d}y \exp(-\pi y/2)}{y^{1/2}(y^2+\frac{1}{4}) \cosh \pi y}$$
$$\times \exp\left(-\frac{1}{96y^2} \frac{1}{|g|^2}\right)[1+\mathrm{O}(g^2)] \tag{3.8}$$

where again the upper (lower) sign is taken if $\arg g = \pi/2\ (-\pi/2)$. The integral in (3.8) may be evaluated by steepest descent for small $|g|$ and yields

$$\lim_{n\to 0} \frac{1}{n} \operatorname{Im} Z_n(g) = \mp 2(9\pi)^{1/6}(2\pi)^{1/2}|g|^{4/3} \exp\left(-\frac{1}{24|g|^2} - \frac{(9\pi)^{2/3}}{8|g|^{2/3}}\right)[1+\mathrm{O}(|g|^{2/3})] \tag{3.9}$$

up to exponentially smaller terms. This result should be compared with (2.21) which was obtained by assuming that the set of dominant terms in the sum when $n = 0$ could be extracted from the sum even though the number of terms in the sum was vanishing. The expression (3.9) is more complicated; in particular, the $\mathrm{O}(|g|^{-2/3})$ in the exponent comes from the continuation of the $(\pm\mathrm{i})^{r}\,{}^{n+1}C_r$ factors. The possibility that this factor is modified by higher-order terms is discussed in § 5.

From the more detailed study of the structure of $Z_n(g)$ carried out in the appendix it is clear that $\lim_{n\to 0} (1/n)[Z_n(g^2)-1] \to$ constant as $|g^2| \to \infty$. Therefore, we cannot use a straightforward dispersion relation, as in the positive integer case, to obtain the asymptotic behaviour of the perturbation expansion, since the contour at infinity cannot be discarded. Instead we have to use a once subtracted dispersion relation. Suppose $\lim_{n\to 0} (1/n)[Z_n(g^2)-1]$ is analytic apart from a branch cut along the negative real g^2 axis; then consideration of the contour shown in figure 3 yields

$$\lim_{n\to 0} \frac{1}{n}[Z_n(g^2)-1] - \lim_{n\to 0} \frac{1}{n}[Z_n(\alpha g^2)-1]$$
$$= \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}g'^2 \lim_{n\to 0} \frac{1}{n}[Z_n(g'^2)-1]\left(\frac{1}{g'^2-g^2} - \frac{1}{g'^2-\alpha g^2}\right)$$
$$= (1-\alpha)g^2 \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}g'^2 \lim_{n\to 0} \frac{1}{n}[Z_n(g'^2)-1] \frac{1}{(g'^2-g^2)(g'^2-\alpha g^2)} \tag{3.10}$$

where $\alpha \neq 1$ is some real positive number. Since $\lim_{n\to 0} [Z_n(g'^2)-1] \to$ constant as $|g'^2| \to \infty$ the contour at infinity may be discarded and one finds

$$\lim_{n\to 0} \frac{1}{n}[Z_n(g^2)-1] - \lim_{n\to 0} \frac{1}{n}[Z_n(\alpha g^2)-1]$$
$$= \frac{(1-\alpha)g^2}{\pi} \int_{-\infty}^0 \frac{\lim_{n\to 0}(1/n) \operatorname{Im} Z_n(g'^2; \arg g' = \pi/2)\, \mathrm{d}g'^2}{(g'^2-g^2)(g'^2-\alpha g^2)} \qquad g^2>0, \tag{3.11}$$

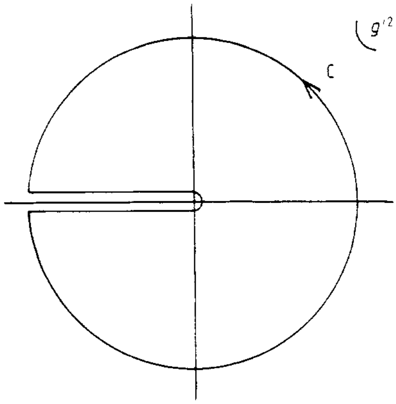

Figure 3. Contour involved in the dispersion relation for $n=0$.

where we have used the fact that the imaginary parts for $\arg g'^2 = \pm\pi$ are equal in magnitude but opposite in sign. Comparing powers of g^{2K} in the perturbative expansion of (3.11), we obtain

$$\lim_{n\to 0}\frac{1}{n}[Z_K(n)] = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\lim_{n\to 0}(1/n)\,\mathrm{Im}\,Z_n(g'^2;\arg g' = \pi/2)\,\mathrm{d}g'^2}{(g'^2)^{K+1}},$$
$$K = 1, 2, \ldots, \tag{3.12}$$

exactly the same result we would have found if a subtraction had not been necessary. Substituting (3.9) into (3.12) and performing a steepest descent calculation for large K yields

$$\lim_{n\to 0}\frac{1}{n}[Z_K(n)] = \frac{1}{(72\pi^2)^{1/6}}K!(-24)^K K^{-5/3}$$
$$\times \exp[-\tfrac{3}{4}(9\pi^2)^{1/3}K^{1/3}]\left[1 + \mathrm{O}\left(\frac{1}{K^{1/3}}\right)\right]. \tag{3.13}$$

This form differs from the one usually found for the asymptotic behaviour of quantities calculated perturbatively in scalar theories because of the appearance of the exp($-$constant $K^{1/3}$) factor. This is a direct consequence of our continuation to $n=0$. The most important feature, however, is still the oscillatory nature of the series for large K which means that the zero-dimensional theory is well defined for real g. Having studied the zero-dimensional case in considerable detail we now go on and apply the same techniques to the field theory in $d = 6-\varepsilon$ dimensions.

4. The $n=0$ limit of the d-dimensional field theory

The starting point for the discussion of this section is the Hamiltonian (2.1) without the quartic and higher-order terms:

$$\mathscr{H}(\phi_i) = \int \mathrm{d}^d x[\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i) + \tfrac{1}{2}r_0\phi_i\phi_i + (g/3)\rho_{ijk}\phi_i\phi_j\phi_k]. \tag{4.1}$$

The saddle points in the combined function space and internal (n-dimensional) space—the instantons $\phi_{i,c}(x)$—satisfy

$$-\nabla^2\phi_i + r_0\phi_i + g\rho_{ijk}\phi_j\phi_k = 0. \tag{4.2}$$

For positive integer n and real coupling constant g, the solutions of least action take the form

$$\phi_{i,c}(x) = u_i\phi_c(x). \tag{4.3}$$

To prove this consider the general solution written as $\phi_{i,c}(x) = u_i(x)\phi_c(x)$ and compute its action. It equals

$$\begin{aligned}\mathcal{H}(\phi_{i,c}) &= \tfrac{1}{2}\int d^dx(\phi_c)^2\Big(\sum_i(\nabla u_i)^2\Big)\\ &\quad+\int d^dx[\tfrac{1}{2}u_iu_i\phi_c(-\nabla^2+r_0)\phi_c+(g/3)\rho_{ijk}u_iu_ju_k\phi_c^3]\\ &\geqslant \int d^dx[\tfrac{1}{2}u_iu_i\phi_c(-\nabla^2+r_0)\phi_c+(g/3)\rho_{ijk}u_iu_ju_k\phi_c^3]\end{aligned} \tag{4.4}$$

with equality only when $\nabla u_i = 0$ for all i. Assuming a solution of the form (4.3), we see from (4.2) that u_i is proportional to $\rho_{ijk}u_ju_k$ for all i. Since the u_i are independent of x the constant of proportionality is x independent and because we are free to choose the normalisation of the u_i we can take it to be unity. Thus (4.2) reduces to

$$u_i = \rho_{ijk}u_ju_k, \tag{4.5a}$$

$$-\nabla^2\phi_c + r_0\phi_c + g\phi_c^2 = 0. \tag{4.5b}$$

From the discussion of the continuation to $n=0$ in §§ 2 and 3 we would expect the dominant contribution to the imaginary parts of the Green functions of the $n=0$ field theory to come from solutions of the type (4.3). However, we have no proof of this and so for the moment (4.3) must remain an ansatz as far as the continuation to $n=0$ is concerned.

The problem of finding the saddle points has now been reduced to solving the two equations (4.5), both of which have previously been studied. The equations for the u_i were solved in § 2 and the equation for $\phi_c(x)$ is identical to the saddle point equation for a one-component ϕ^3 theory which was considered by McKane (1979). An analytic form for $\phi_c(x)$ can be found in one dimension; however, in three dimensions, for instance, the solution has to be found numerically. This is not as bad as it seems since it is known that the solutions of least action are spherically symmetric and thus (4.5*b*) reduces to an ordinary differential equation.

In the rest of this section our discussion will be limited to the case where $d=6-\varepsilon$ for two main reasons. Firstly, calculations near to, or in, six dimensions can be performed analytically and the method can therefore be more clearly illustrated. Secondly, we eventually want to characterise the asymptotic behaviour of the ε expansion, and calculating the asymptotic behaviour of the perturbation expansion in g and in $d=6-\varepsilon$ dimensions is a necessary prerequisite. The philosophy and technical aspects of carrying out instanton calculations using dimensional regularisation is discussed by McKane and Wallace (1978) and McKane (1979) (see also Drummond and Shore (1979) for a slightly different approach). The instanton solution relevant to $d=6$ is used; any corrections due to the fact we are working in $6-\varepsilon$ dimensions are treated perturbatively in ε. In exactly six dimensions the required instanton with least action is a solution of (4.5b) with $r_0=0$, is spherically symmetric and is given by

$$\phi_c(x; x_0, \lambda) = -\frac{24}{g}\frac{\lambda^2}{[1+\lambda^2(x-x_0)^2]^2} \tag{4.6}$$

where x_0 and λ are the position and (inverse) size of the instanton respectively. For the solutions of least action we therefore find, using (4.5),

$$\begin{aligned}\mathcal{H}(u_i^{(r)}\phi_c) &= -\tfrac{1}{6}u_i^{(r)}u_i^{(r)}\int \mathrm{d}^d x\,(g\phi_c^3)\\ &= \frac{r(n+1-r)}{(n+1)^2(n+1-2r)^2}\frac{192\pi^3\lambda^\varepsilon}{5g^2}[1+\mathrm{O}(\varepsilon)].\end{aligned} \tag{4.7}$$

Here we are primarily interested in the Green functions of the field theory, and not the partition function which was used for illustration in earlier sections. The N-point Green function is defined by

$$G^{(N)}_{i_1 i_2\ldots i_N}(x_1, x_2, \ldots, x_N) = \frac{\int \mathrm{D}\boldsymbol{\phi}\; \phi_{i_1}(x_1)\phi_{i_2}(x_2)\ldots\phi_{i_N}(x_N)\exp -\mathcal{H}(\boldsymbol{\phi})}{\int \mathrm{D}\boldsymbol{\phi}\exp -\mathcal{H}(\boldsymbol{\phi})}. \tag{4.8}$$

Performing a saddle point evaluation of (4.8) using the solutions discussed above, it is easy to see that the leading asymptotic behaviour of (4.8) is found by using the perturbative ($r=0$) solution in the denominator and the non-perturbative ($r>0$) solutions in the numerator. We therefore obtain

$$\begin{aligned}&G^{(N)}_{i_1 i_2\ldots i_N}(x_1, x_2, \ldots, x_N)\\ &\quad= \sum_{\text{all solutions}} u_{i_1}u_{i_2}\ldots u_{i_N}\phi_c(x_1)\phi_c(x_2)\ldots\phi_c(x_N)\exp -\mathcal{H}(\boldsymbol{\phi}_c)\\ &\qquad\times\left(\frac{\det M}{\det M_0}\right)^{-1/2} C_1[1+\mathrm{O}(g^2)]\end{aligned} \tag{4.9}$$

where C_1 is a factor which arises because (4.6) is not a solution of the saddle point equation (4.5b) in $6-\varepsilon$ dimensions (McKane 1979) and where

$$M_{ij} = -\nabla^2\delta_{ij} - 48\rho_{ijk}u_k\lambda^2/[1+\lambda^2(x-x_0)^2]^2 \tag{4.10a}$$

and

$$M_{0,ij} = -\nabla^2\delta_{ij}. \tag{4.10b}$$

Since our purpose is to investigate the renormalisation group functions we will only be interested in the $N=2$ and $N=3$ Green functions. Now

$$\sum_{\text{all solutions}} u_{i_1}u_{i_2} = \frac{\delta_{i_1 i_2}}{n}\sum_r {}^{n+1}C_r u_i^{(r)}u_i^{(r)} \tag{4.11a}$$

and

$$\sum_{\text{all solutions}} u_{i_1}u_{i_2}u_{i_3} = \frac{\rho_{i_1 i_2 i_3}}{n\alpha(n)}\sum_r {}^{n+1}C_r u_i^{(r)}u_i^{(r)} \tag{4.11b}$$

where $\alpha(n)$ is defined by $\rho_{ilm}\rho_{jlm} = \alpha(n)\delta_{ij}$. From now on we therefore omit the $\delta_{i_1 i_2}$ and $\rho_{i_1 i_2 i_3}/\alpha(n)$ factors and $G^{(N)}$ will be taken to mean $G^{(2)}$ or $G^{(3)}$, although the calculational method goes through with little change to the case of general N. The small oscillations determinant M can be diagonalised in (i, j) space using the eigenvectors (2.12), and (4.9) becomes

$$G^{(N)}(x_1, x_2, \ldots, x_N)$$
$$= \frac{1}{n} \sum_r {}^{n+1} C_r u_i^{(r)} u_i^{(r)} \phi_c(x_1) \phi_c(x_2) \ldots \phi_c(x_N) \exp -\mathscr{H}(\phi_c) C_1$$
$$\times \left(\frac{\det M_{(1)}}{\det M_{(0)}}\right)^{-1/2} \left(\frac{\det M_{(2)}}{\det M_{(0)}}\right)^{-(n-r)/2}$$
$$\times \left(\frac{\det M_{(3)}}{\det M_{(0)}}\right)^{-(r-1)/2} [1+\mathrm{O}(g^2)] \tag{4.12}$$

where

$$M_{(a)} = -\nabla^2 - \frac{4\zeta_{(a)}(\zeta_{(a)}+1)\lambda^2}{[1+\lambda^2(x-x_0)^2]^2}, \qquad a = 0, 1, 2, 3, \tag{4.13}$$

with

$$\zeta_{(a)}(\zeta_{(a)}+1) = \begin{cases} 0, & a=0, \\ 12, & a=1, \\ -12r/(n+1-2r), & a=2, \\ 12(n+1-r)/(n+1-2r), & a=3. \end{cases} \tag{4.14}$$

The eigenfunctions and eigenvalues of $M_{(1)}$, $M_{(2)}$ and $M_{(3)}$ are not known analytically in d dimensions, and thus $\det M_{(a)}/\det M_{(0)}$ cannot be calculated directly. However, this combination can be calculated indirectly by mapping the theory onto a hypersphere in $d+1$ dimensions. One then finds (McKane 1979) that

$$\frac{\det M_{(a)}}{\det M_{(0)}} = \frac{\det V_{(a)}}{\det V_{(0)}}$$
$$= \prod_{L=0}^{\infty} \left(\frac{(L+d/2+\zeta_{(a)})(L+d/2-\zeta_{(a)}-1)}{(L+d/2)(L+d/2-1)}\right)^{\nu_L}, \qquad a = 1, 2, 3, \tag{4.15}$$

where

$$V_{(a)} = -L^2 - (d/2)(d/2-1) + \zeta_{(a)}(\zeta_{(a)}+1), \qquad a = 0, 1, 2, 3, \tag{4.16}$$

and

$$\nu_L = \frac{\Gamma(L+d-1)(2L+d-1)}{\Gamma(d)\Gamma(L+1)}. \tag{4.17}$$

L^2 in (4.16) is the square of the angular momentum operator in $d+1$ dimensions and ν_L is the degeneracy of the eigenvalues of L^2 in this dimension. Examination of (4.15) and (4.17) when $a=1$ shows that the $L=0$ mode gives a negative contribution (and therefore an imaginary contribution to (4.12)), the $(d+1)$ $L=1$ modes give $\mathrm{O}(\varepsilon)$ contributions (leading to possible divergences in (4.12)) and the $L>1$ modes are harmless. The ratio of determinants when $a=1$ is identical to that found in the one-component case. Rather than repeat the discussion given in that case, we merely recall that to cope with the potential zero modes, caused by the breaking of translational and dilatational invariance by the solution (4.6), collective coordinates have to be introduced. Therefore (4.12) should read

$$G^{(N)}(x_1, x_2, \ldots, x_N)$$
$$= \frac{1}{n}\sum_r {}^{n+1}C_r u_i^{(r)} u_i^{(r)} \int \mathrm{d}^d x_0 \, \mathrm{d}\lambda \, J^V (2\pi)^{-(d+1)/2} \phi_c(x_1)\phi_c(x_2)\ldots\phi_c(x_N)$$
$$\times \exp -\mathcal{H}(\phi_c) C_1 \left(\frac{\det \tilde{V}_{(1)}}{\det V_{(0)}}\right)^{-1/2} \left(\frac{\det V_{(2)}}{\det V_{(0)}}\right)^{-(n-r)/2}$$
$$\times \left(\frac{\det V_{(3)}}{\det V_{(0)}}\right)^{-(r-1)/2} [1+\mathrm{O}(g^2)] \tag{4.18}$$

where J^V is the Jacobian of the transformation to collective coordinates, the $(2\pi)^{-1/2}$ factors come from the absence of $(d+1)$-Gaussian integrals in the numerator and the tilde indicates that the potential zero modes have been extracted.

It is not our purpose here to evaluate the various terms in (4.18) in detail; we are more interested in discussing the continuation to $n=0$. To this end let us separate out various terms in (4.18) which bring out the analogy with the $d=0$ case. In particular let us note from (4.15) that the $L=0$ contribution from the product of the determinants is (setting $\varepsilon=0$)

$$(2\pi)^{-n/2}\left(\frac{2\pi}{-1}\right)^{1/2}\left(\frac{2\pi(n+1-2r)}{(n+1)}\right)^{(n-r)/2}\left(\frac{2\pi(n+1-2r)}{-(n+1)}\right)^{(r-1)/2} \tag{4.19}$$

which should be compared with (2.13). It is clear that the continuation to $n=0$ will go through in the same way as for the $d=0$ case, and from (4.15) it can be seen that there is no possibility of zero or negative modes when $n=0$, $r\to\infty$ and $L>0$ (apart from the translational and dilatational modes already extracted). Let us now go through the continuation rather more carefully by writing

$$G^{(N)}(x_1, x_2, \ldots, x_N)$$
$$= \int \mathrm{d}\lambda \, \mathrm{d}^d x_0 \frac{1}{n}\sum_{r=1}^{[n/2]} {}^{n+1}C_r \left(\frac{2\pi}{-1}\right)^{1/2}\left(\frac{2\pi(n+1-2r)}{(n+1)}\right)^{(n-r)/2}$$
$$\times \left(\frac{2\pi(n+1-2r)}{-(n+1)}\right)^{(r-1)/2} \exp\left(\frac{-r(n+1-r)}{(n+1)^2(n+1-2r)^2}\,\frac{192\pi^3\lambda^\varepsilon}{5g^2}\right)$$
$$\times \gamma^{(N)}(x_1, x_2, \ldots, x_N; r, n)[1+\mathrm{O}(g^2, \varepsilon)] \tag{4.20}$$

where

$$\gamma^{(N)}(x_1, x_2, \ldots, x_N; r, n)$$
$$= u_i^{(r)} u_i^{(r)} \phi_c(x_1)\phi_c(x_2)\ldots\phi_c(x_N) J^V (2\pi)^{-(d+1)/2}(2\pi)^{-n/2} C_1$$
$$\times \tilde{\prod_{L=1}} \left(\frac{(L+3+d/2)(L-4+d/2)}{(L+d/2)(L+d/2-1)}\right)^{-\nu_L/2}$$
$$\times \prod_{L=1}^{\infty} \left(\frac{(L+d/2+\zeta_{(2)})(L+d/2-\zeta_{(2)}-1)}{(L+d/2)(L+d/2-1)}\right)^{-(n-r)\nu_L/2}$$
$$\times \prod_{L=1}^{\infty} \left(\frac{(L+d/2+\zeta_{(3)})(L+d/2-\zeta_{(3)}-1)}{(L+d/2)(L+d/2-1)}\right)^{-(r-1)\nu_L/2}. \tag{4.21}$$

Concentrating on (4.20) for the moment, we see a very similar expression to (2.13). From our experience in §§ 2 and 3, we expect that the correct signs for the factors of i should be as in (2.14). We can now repeat the analysis of § 3 and arrive at the analogous expression to (3.9). Of course, in this case in order to obtain a meaningful expression the theory has to be renormalised. This should present no difficulties; renormalisation of similar expressions has recently been carried out (McKane *et al* 1984, Newlove 1984), but it is not our intention here to discuss this rather technical problem and we restrict ourselves to making a few general comments in § 5. Luckily we do not have to go through the renormalisation process to obtain the dominant terms in the asymptotic behaviour of $G^{(N)}$: if $G_K^{(N)}$ is the coefficient of g^{2K} then it follows from the methods of § 3 that for large K

$$G_K^{(N)} \sim K!(-5/48\pi^3)^K. \tag{4.22}$$

This structure will appear in all renormalisation group functions as well as in the Green functions.

Corrections to (4.22) can be computed systematically from (4.20) and (4.21) after renormalisation. The expression (4.21) was separated out since it does not contain terms which affect the continuation to $n=0$. There are zero and negative modes for positive n in the determinant factors in (4.21) and one has to imagine a regulator controlling these while the limit $n \to 0$ is being taken. The net effect, however, is merely to set $n=0$ and $r = O(|g|^{-2/3})$ in (4.21).

5. Conclusions

In this paper we have discussed a systematic method of investigating the asymptotic behaviour of the perturbation expansions for $n=0$ cubic theories, although the method should also be applicable to other $n=0$ interactions (the well known example of the $O(n)$ invariant $(\boldsymbol{\phi}^2)^2$ interaction does not need this treatment since all the solutions labelled by the group parameters have the same action and thus there is no equivalent of the sum on r). The approach was illustrated for the percolation problem where the solutions controlling the asymptotic behaviour were labelled by the integer $r = 1, 2, \ldots, n$ for positive integer n. After continuation to $n=0$, it was found that it was the $r=\infty$ (or more precisely the $r = O(|g|^{-2/3})$, g small) solution which characterised the asymptotic behaviour. This corresponded to the solution of greatest action amongst the class of solutions under consideration. This, together with the occurrence of $(n-1)$ potentially massless modes (actually with mass $O(|g|^{2/3})$), is very reminiscent of features found in another $n=0$ problem (Bray and Moore 1978, 1979). Our results disagree with HRW who, for example, find the result $-15/128\pi^3$ in (4.22) (with our normalisation) where we find $-5/48\pi^3$. Moreover we expect that our method is capable of calculating corrections to (4.22).

There are two aspects of the calculation described in this paper that need further investigation.

(i) Renormalisation of the field theory has not been carried out. We expect that this would follow the path laid down by McKane *et al* (1984) and Newlove (1984). The renormalisation scheme would be an extended form of minimal subtraction, but of course the asymptotic behaviour of the ε expansion (determined from that of the renormalisation group functions) would be universal and independent of the details of the renormalisation scheme.

(ii) There are $(n-1)$ nearly massless modes both in $d=0$ and in higher dimensions (they are $L=0$ modes in the field theory). Since they have a mass which is $O(|g|^{2/3})$ they could enter into higher-order corrections in such a way as to cause the breakdown of the perturbation expansion about the saddle point. This is a very difficult question to answer since the factors of r generated from tensor contractions themselves become g dependent, but it is probable that prefactors are changed by these modes and it may be that the $O(|g|^{-2/3})$ term in the exponent is modified or cancelled completely. The correct procedure would be to extract these modes by the introduction of collective coordinates, but since it is not clear what symmetry, if any, has been broken it is not obvious how to carry this out.

The dominant behaviour displayed in (4.22) is not affected by (i) and (ii) above, which only affect the corrections to this result for large K. As it stands, (4.22) contains the most important information for the resummation of the ε expansion in the percolation problem. This resummation together with the application of this method to spin glasses is now under consideration.

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Appendix

In § 2 the saddle points of $Z_n(g)$, the zero-dimensional partition function, were enumerated and the evaluation of the various contributions made by these saddle points to the integral defining $Z_n(g)$ led to the expression (2.13). None of the subtleties of a proper steepest descent calculation were evident in this approach. Here we carry out a more careful analysis of the zero-dimensional case which is designed to show that the crude 'instanton' approach adopted in the main text gives the correct results.

For motivation consider the $n=0$ $(\boldsymbol{\phi}^2)^2$ theory. The zero-dimensional partition function is

$$\bar{Z}_n(g)=\int \mathrm{d}\boldsymbol{\phi}\, \exp[-\tfrac{1}{2}\boldsymbol{\phi}^2-(g/4)(\boldsymbol{\phi}^2)^2]. \tag{A1}$$

This presents no difficulties since if we go over to spherical polar coordinates in n dimensions we can perform the angular integrals trivially:

$$\bar{Z}_n(g)=S_n\int_0^\infty \mathrm{d}R\, R^{n-1}\exp(-\tfrac{1}{2}R^2-gR^4/4) \tag{A2}$$

where $R^2=\boldsymbol{\phi}^2$ and S_n is the surface area of a sphere in n dimensions. The integral in (A2) can now be studied carefully using steepest descent methods. In this case the $n=0$ limit is trivial.

The analogous expression for the theory with the symmetry of the $(n+1)$-state Potts model, as given by (2.5), is

$$Z_n(g)=S_n\int_0^\infty \mathrm{d}R\, R^{n-1}\exp(-\tfrac{1}{2}R^2)f_n(gR^3) \tag{A3}$$

where

$$f_n(gR^3)=\int \mathrm{d}\Omega_n \exp[(-g/3)\rho_{ijk}\phi_i\phi_j\phi_k]\left(\int \mathrm{d}\Omega_n\right)^{-1} \tag{A4}$$

and where Ω_n is the solid angle in n dimensions. The purpose of this appendix is to study the form of $f_n(gR^3)$ for various values of n ($n=2$, $n=3$, general positive integer n, $n=0$) and then evaluate $Z_n(g)$ from (A3) by steepest descent. We begin with $n=2$ and $n=3$ where $f_n(gR^3)$ can be evaluated in terms of familiar special functions.

$n=2$

Taking the following representation for $\{e_i^\alpha\}$,

$$e^1=\frac{1}{\sqrt{2}}(0,2),\qquad e^2=\frac{1}{\sqrt{2}}(\sqrt{3},-1),\qquad e^3=\frac{1}{\sqrt{2}}(-\sqrt{3},-1), \tag{A5}$$

the only non-zero ρ_{ijk} are $\rho_{111}=3/\sqrt{2}$ and $\rho_{112}=-3/\sqrt{2}$. Therefore,

$$\rho_{ijk}\phi_i\phi_j\phi_k=(3/\sqrt{2})(\phi_2^3-3\phi_1^2\phi_2). \tag{A6}$$

With $\phi_1=R\sin\chi$ and $\phi_2=R\cos\chi$ one finds that

$$f_2(gR^3)=\frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\exp(-gR^3\cos\theta/\sqrt{2})$$
$$=I_0(gR^3/\sqrt{2}) \tag{A7}$$

where I_0 is the usual modified Bessel function (Watson 1944). It has the following properties of interest to us:

$$I_0(z)=\sum_{K=0}^{\infty}\frac{(z^2/4)^K}{(K!)^2} \tag{A8}$$

$$I_0(z)=\begin{cases}\dfrac{1}{(2\pi z)^{1/2}}(\mathrm{e}^z+\mathrm{i}\mathrm{e}^{-z})\left[1+\mathrm{O}\left(\dfrac{1}{z}\right)\right], & |z| \text{ large, } 0\leqslant\arg z\leqslant\pi, \quad \text{(A9}a\text{)}\\[2ex] \dfrac{1}{(2\pi z)^{1/2}}(\mathrm{e}^z-\mathrm{i}\mathrm{e}^{-z})\left[1+\mathrm{O}\left(\dfrac{1}{z}\right)\right], & |z| \text{ large, } 0\geqslant\arg z\geqslant-\pi. \quad \text{(A9}b\text{)}\end{cases}$$

Now if one began from the $n=2$ version of (2.5) and performed a proper steepest descent calculation, one would presumably find a sudden change in the steepest descent contours for $\arg g=0$ reflecting the presence of a branch cut. In this approach the change in the steepest descent contours shows up as a Stokes phenomenon: the coefficient of e^{-z} in (A9) changes discontinuously at $\arg z=0$. As explained in § 2, one either works with the partition function defined for $0\leqslant\arg g^2\leqslant 2\pi$, in which case (A9*a*) is appropriate, or with the partition function defined for $0\geqslant\arg g^2\geqslant-2\pi$, in which case (A9*b*) is appropriate. Here we choose to work with the former. Using (A9*a*) the steepest descent evaluation of $Z_2(g)$ can now be carried out. For real g

there are two saddle points: $R=0$ and $R=O(g^{-1})$. To investigate the latter it is sufficient to use the asymptotic form

$$f_2(gR^3)=\frac{1}{(2\pi|g|R^3)^{1/2}}\exp\left(\frac{|g|R^3}{\sqrt{2}}\right)[1+O(g^{-1}R^{-3})], \qquad g \text{ real}, \tag{A10}$$

since corrections near the saddle point are $O(g^{-1}R^{-3})=O(g^2)$. The expression (A10) is valid if $\arg g=0$ or $\arg g=\pi$. A calculation along similar lines to the one-component case (McKane 1979) now yields

$$\operatorname{Im} Z_2(g)=\pm\sqrt{3}\pi\exp(-1/27g^2)[1+O(g^2)] \tag{A11}$$

where the upper sign is taken when $\arg g=0$ and the lower sign when $\arg g=\pi$. Using the dispersion relation technique discussed in § 2, the coefficient of g^{2K} in the expansion of $Z_2(g)$ in powers of g^2 is found from (A11) to be

$$Z_K(2)=\sqrt{3}(27)^K K!\,K^{-1}[1+O(1/K)] \tag{A12}$$

for large K. This shows the expected non-oscillatory behaviour for real g. This result may be checked directly by substituting (A8) into (A3) and integrating.

n = 3

The exact treatment for this value of n can be made simpler by making the following choice for $\{e_i^\alpha\}$:

$$\begin{aligned} &e^1=(1,1,1), &&e^2=(1,-1,-1),\\ &e^3=(-1,1,-1), &&e^4=(-1,-1,1). \end{aligned} \tag{A13}$$

The only non-zero ρ_{ijk} is $\rho_{123}=4$ and therefore

$$\rho_{ijk}\phi_i\phi_j\phi_k=24\phi_1\phi_2\phi_3. \tag{A14}$$

Going over to spherical polar coordinates one finds that

$$f_3(gR^3)=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\chi\,\mathrm{d}\theta\sin\theta\exp(-8gR^3\sin^2\theta\cos\theta\cos\chi\sin\chi) \tag{A15}$$

$$=\frac{\pi}{3}\sum_{K=0}^{\infty}\left(\frac{2gR^3}{3\sqrt{3}}\right)^{2K}\frac{\Gamma(2K+1)}{\Gamma(K+\frac{5}{6})\Gamma^2(K+1)\Gamma(K+\frac{7}{6})} \tag{A16}$$

$$=\tfrac{1}{3}\pi I_{1/6}(4gR^3/3\sqrt{3})I_{-1/6}(4gR^3/3\sqrt{3}). \tag{A17}$$

The asymptotic expansions for $I_{\pm1/6}(z)$ then give

$$f_3(gR^3)=\frac{1}{8\sqrt{3}gR^3}\left[\exp\left(\frac{8gR^3}{3\sqrt{3}}\right)\pm i\sqrt{3}-\exp\left(-\frac{8gR^3}{3\sqrt{3}}\right)\right][1+O(g^{-1}R^{-3})] \tag{A18}$$

where the sign of the constant term depends on the value of $\arg g$ just as in (A9). We again choose to take the positive sign. The steepest descent evaluation of $Z_3(g)$ now proceeds in an exactly analogous way to that of $Z_2(g)$ and gives

$$\operatorname{Im} Z_3(g)=\pm(2\pi)^{3/2}\exp(-1/128g^2)[1+O(g^2)] \tag{A19}$$

where the signs are chosen as in (A11). Using (2.18), the coefficient of g^{2K} is found to be

$$Z_K(3)=2(2\pi)^{1/2}(128)^K K!\,K^{-1}[1+O(1/K)] \tag{A20}$$

for large K. Once again we have an exact result, (A16), against which our method for obtaining Z_K can be checked.

General positive integer n

To find Im $Z_n(g)$ using this approach we need to find the generalisation of (A9) and (A18) to general n. Let us begin by obtaining a recurrence relation for f_n. Suppose $\{E_I^A;\, A=1,2,\ldots,n+2,\, I=1,2,\ldots,n+1\}$ is the set of Potts vectors in $(n+1)$ dimensions. They may be written in terms of the Potts vectors in n dimensions $\{e_i^\alpha;\, \alpha=1,2,\ldots,n+1,\, i=1,2,\ldots,n\}$ as follows:

$$\boldsymbol{E}^\alpha=(n+1)^{-1/2}((n+2)^{1/2}e_i^\alpha,-1), \qquad \boldsymbol{E}^{n+2}=(n+1)^{-1/2}(0,0,\ldots,0,n+1). \tag{A21}$$

If we write

$$P_{IJK}=\sum_{A=1}^{n+2} E_I^A E_J^A E_K^A \qquad \text{and} \qquad \boldsymbol{\Phi}=(\phi_1,\phi_2,\ldots,\phi_n,\phi_{n+1}) \tag{A22}$$

then it follows that

$$P_{IJK}\Phi_I\Phi_J\Phi_K=\left(\frac{n+2}{n+1}\right)^{3/2}\rho_{ijk}\phi_i\phi_j\phi_k-\frac{3(n+2)}{(n+1)^{1/2}}\phi_i\phi_i\phi_{n+1}+\frac{n(n+2)}{(n+1)^{1/2}}\phi_{n+1}^3. \tag{A23}$$

This enables us to relate f_{n+1} to f_n:

$$\begin{aligned} f_{n+1}(gR^3)=&\int_0^\pi \sin^{n-1}\theta\,\mathrm{d}\theta\, f_n\left[g\left(\frac{n+2}{n+1}\right)^{3/2}R^3\sin^3\theta\right]\\ &\times\exp\left(\frac{(n+2)}{(n+1)^{1/2}}gR^3\sin^2\theta\cos\theta\right.\\ &\left.-\frac{g}{3}\frac{n(n+2)}{(n+1)^{1/2}}R^3\cos^3\theta\right)\left(\int_0^\pi\sin^{n-1}\theta\,\mathrm{d}\theta\right)^{-1}. \end{aligned} \tag{A24}$$

The asymptotic expansion for $f_n(gR^3)$ is found to be

$$\begin{aligned} f_n(gR^3)=&\frac{(2\pi)^{(n-1)/2}}{S_n}\sum_{r=1}^{n}(\pm\mathrm{i})^{r-1}\,{}^{n+1}C_r\frac{r^{(n-1)/4}(n+1-r)^{(n-1)/4}}{(n+1)^{n-1}}\frac{1}{(gR^3)^{(n-1)/2}}\\ &\times\exp\left(\frac{gR^3}{3}\frac{(n+1)(n+1-2r)}{(n+1-r)^{1/2}r^{1/2}}\right)[1+\mathrm{O}(g^{-1}R^{-3})] \end{aligned} \tag{A25}$$

where the signs are chosen as before. This was obtained as follows. The general form is clear from our experience with $n=2$ and 3 and from the knowledge that a steepest descent calculation must give (2.13) up to factors of $\pm\mathrm{i}$ and $\frac{1}{2}$. Use of the recurrence relation (A24) determines these factors. The verification of (A25) involves rather a lot of tedious algebra which we omit. Choosing the plus signs in (A25) as usual and noting that when arg $g=0$ the $r=1,2,\ldots,[n/2]$ terms give the dominant contributions but that when arg $g=\pi$ the $r=n, n-1,\ldots,n+1-[n/2]$ terms are the important ones, we find

$$\begin{aligned} f_n(gR^3)=&\frac{(2\pi)^{(n-1)/2}}{S_n}\sum_{r=1}^{[n/2]}(\pm\mathrm{i})^{r-1}\,{}^{n+1}C_r\frac{r^{(n-1)/4}(n+1-r)^{(n-1)/4}}{(n+1)^{n-1}}\frac{1}{(|g|R^3)^{(n-1)/2}}\\ &\times\exp\left(\frac{|g|R^3}{3}\frac{(n+1)(n+1-2r)}{(n+1-r)^{1/2}r^{1/2}}\right)[1+\mathrm{O}(g^{-1}R^{-3})] \end{aligned} \tag{A26}$$

where g is real, the plus sign referring to arg $g=0$ and the minus sign to arg $g=\pi$. Using (A3) and (A26) and performing similar calculations to those described for $n=2$ and $n=3$ gives the result (2.14).

It should be noted that including a factor S_n in the definition of f_n in (A4) ensures that it has a non-zero limit as $n \to 0$. For example, the first term in the perturbation expansion for general n is

$$f_n(gR^3) = 1 + \frac{g^2R^6(n+1)^2(n-1)}{3(n+2)(n+4)} + \mathrm{O}(g^2R^6)^2. \tag{A27}$$

n = 0

Using the continuation method discussed in § 3, and in particular (3.6), we find that

$$f_0(gR^3) = \frac{\pi}{2\mathrm{i}} \int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty} \frac{F(z, gR^3)\,\mathrm{d}z}{\sin^2 \pi z} \tag{A28}$$

where we have used $S_n = 2\pi^{n/2}/\Gamma(n/2) = n + \mathrm{O}(n^2)$ as $n \to 0$ and where

$$F(z, gR^3) = (2\pi)^{-1/2}(gR^3)^{1/2} \exp[\pm \mathrm{i}\pi(z-1)][z(1-z)]^{-1/4}[z(1-z)\pi \operatorname{cosec} \pi z]^{-1}$$
$$\times \exp\left(\frac{gR^3}{3} \frac{(1-2z)}{[z(1-z)]^{1/2}}\right)[1 + \mathrm{O}(g^{-1}R^{-3})]. \tag{A29}$$

Unlike the treatment in § 3, this result was obtained without restricting ourselves to n even and without taking g to be pure imaginary (the signs are as before: the upper sign is taken if $0 \leq \arg g \leq \pi$ and the lower sign is taken if $0 \geq \arg g \geq -\pi$). Taking $\delta = \frac{1}{2}$ without loss of generality and writing $z = \frac{1}{2} + \mathrm{i}y$ one finds

$$f_0(gR^3) = \frac{(\mp \mathrm{i}gR^3)^{1/2}}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y\, \mathrm{e}^{\pi y/2}}{(y^2+\frac{1}{4})^{5/4} \cosh \pi y}$$
$$\times \exp\left(-\frac{2y}{3} \frac{(\mp \mathrm{i}gR^3)}{(y^2+\frac{1}{4})^{1/2}}\right)[1 + \mathrm{O}(g^{-1}R^{-3})]. \tag{A30}$$

Just as the (appropriately defined) one-component ϕ^3 integral with a pure imaginary coupling constant has no non-trivial saddle points, the expression (A30) with g real leads to no non-trivial saddle points when substituted into (A3). Thus for real g the partition function when $n = 0$ is completely described by the perturbation expansion—there is no imaginary part, no discontinuity and no branch cut. On the other hand if g is pure imaginary there is a non-trivial saddle point which leads to an imaginary part for the partition function. From (A30) it can be seen that $f_0(gR^3)$ is real if $\arg g = \pi/2$ and also equal to $f_0(gR^3)$ when $\arg g = -\pi/2$. Performing a steepest descent calculation for large $|gR^3|$ leads to

$$f_0(gR^3) = \frac{(9\pi)^{1/6}2^{1/2}}{(\mp \mathrm{i}gR^3)^{1/6}} \exp\left(\mp \frac{2\mathrm{i}gR^3}{3} - \frac{(9\pi)^{2/3}}{4}(\mp \mathrm{i}gR^3)^{1/3}\right)[1 + \mathrm{O}(g^{-1}R^{-3})] \tag{A31}$$

with the upper sign referring to $\arg g = \pi/2$ and the lower sign to $\arg g = -\pi/2$.

If we write $h = \mathrm{i}g$ then $\arg h = \pi$ when $\arg g = \pi/2$ and $\arg h = 0$ when $\arg g = -\pi/2$. We can then write (A31) in a form analogous to (A10) and (A26):

$$f_0(-\mathrm{i}hR^3) = \frac{(9\pi)^{1/6}2^{1/2}}{(\pm hR^3)^{1/6}} \exp\left(\pm \frac{2hR^3}{3} - \frac{(9\pi)^{2/3}}{4}(\pm hR^3)^{1/3}\right)[1 + \mathrm{O}(h^{-1}R^{-3})] \tag{A32}$$

where the upper sign refers to $\arg h = 0$ and the lower sign to $\arg h = \pi$. The most obvious difference between (A32) and (A10) or (A26) is the occurrence of a term proportional to $(|h|R^3)^{1/3}$ in the exponential. This comes about because of the $e^{\pi y/2}$ sech πy term in (A30) which ensures convergence of the integral for large $|y|$. Substituting (A32) into (A3) (with $R^{n-1}=R^{-1}$) and following the same procedure as for the positive integer n cases, we find

$$\lim_{n\to 0}\left(\frac{1}{n}\operatorname{Im} Z_n(-\mathrm{i}h)\right)$$
$$=\pm 2(9\pi)^{1/6}(2\pi)^{1/2}|h|^{4/3}\exp\left(-\frac{1}{24|h|^2}-\frac{(9\pi)^{2/3}}{8|h|^{2/3}}\right)[1+\mathrm{O}(|h|^{2/3})] \qquad \text{(A33)}$$

where the upper sign is taken when $\arg h=0$ and the lower sign when $\arg h=\pi$. Converting back to the original coupling constant $g=-\mathrm{i}h$ gives

$$\lim_{n\to 0}\left(\frac{1}{n}\operatorname{Im} Z_n(g)\right)$$
$$=\mp 2(9\pi)^{1/6}(2\pi)^{1/2}|g|^{4/3}\exp\left(-\frac{1}{24|g|^2}-\frac{(9\pi)^{2/3}}{8|g|^{2/3}}\right)[1+\mathrm{O}(|g|^{2/3})] \qquad \text{(A34)}$$

where the upper sign is taken when $\arg g=\pi/2$ and the lower sign when $\arg g=-\pi/2$. We expect $\lim_{n\to 0}(1/n)[Z_n(-\mathrm{i}h)-1]$ to be analytic in the upper-half h plane with a cut along the real h axis, in analogy with the positive n integer case. In terms of g, or rather g^2, this means a cut along the negative real g^2 axis. From (A3) and (A30) it is apparent that $\lim_{n\to 0}(1/n)[Z_n(g)-1]\to$ constant as $|g^2|\to\infty$ and thus this analytic structure may be exploited by using a once subtracted dispersion relation to obtain the asymptotic behaviour of the perturbation expansion for the $n=0$ case. This is discussed in more detail in § 3.

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